1. Suppose that m and n are both sums of two integer squares. Show that their product mn is also a sum of two squares. Hint: a number can be written as a sum of squares of integers exactly when it is the norm of something in $\mathbb{Z}[i]$.

2. Try to write the following products as sums of squares. We know it's possible to do this according to Question 1, but it isn't easy. Calculator recommended.

$$5 \times 17 \qquad 17 \times 29 \qquad 5 \times 29$$

3. Suppose that $a^2 + b^2 = c^2$ is a primitive pythagorean triple, which we saw means that a and b are coprime. Write the left hand side of that equation as a norm:

Comparing the sides, we have $N(a + bi) = (a + bi)(a - bi) = c^2$. If these were actual integers and we know that $a + bi$ and $a - bi$ had no common factors, then this would imply that $a + bi$ was already a square to begin with. *This is still true in the complex integers because, among other things, a and b are relatively prime* but with the caveat that we might have to multiply by ± 1 or $\pm i$ to make it a square.

In other words: starting with a PPT, $a^2 + b^2 = c^2$, the left hand side arises as the norm of a complex integer $u(x + yi)^2$ where u can be any of $\pm 1, \pm i$

Let's just fix $u = 1$ for now. Expand $(m + ni)^2$ and collect terms: what you get must some be $a + bi$ associated to a pythagorean triple. What do a and b look like in terms of m and n?

4. Compare your answer to the previous Question to the book's formulas for Pythagorean triples – does it match? (it should match part of it, and we would get the rest by expanding $u(x + iy)$ for the remaining $u = -1, i, -i$).

5. We can also apply Question 1 to Pythagorean triples: if we have two triples

$$a^2 + b^2 = c^2 \qquad d^2 + e^2 = f^2$$

Then Question 1 tells us that we should be able to write $(ef)^2$ as a sum of two squares; in other words if we can find e and f as hypotenuse lengths of an integer triangle, then ef is also the hypotenuse length of some integer triangle.

Test this with the triples

$$3^2 + 4^2 = 5^2 \qquad 5^2 + 12^2 = 13^2$$

In other words find a solution to $x^2 + y^2 = (5 \times 13)^2$. Calculator strongly recommended. If you get stuck/bored, Wikipedia lists solutions on the "Pythagorean Triple" page.

6. We will find all solutions to the Diophantine equation

$$x^2 + y^2 = 5z^2$$

Write this as a statement involving the norm on $\mathbb{Z}[i]$**:**

Your previous answer should have been something like $N(x+yi) = 5z^2$**. In particular, this means that** $N(x + yi)$ **must be divisible by** 5**. Fact: if** $x + iy$ **is divisible by** 5 **then it is divisible by the complex integer** $1 + 2i$**.**

In other words, there is some $a + bi$ **such that** $x + yi = (1 + 2i)(a + bi)$**. Expand the right hand side to express** $x + yi$ **in terms of** a **and** b

We now have the expression

$$5z^2 = N(x + iy) = N((1 + 2i)(a + bi)) = N(1 + 2i)N(a + bi) = 5(a^2 + b^2)$$

What are you left with after dividing the far left and right hand sides by 5**? A familiar kind of number? Summarize your findings.**